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Nonclassical Choices in Variational Principles for Eigenvalues

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1. INTRODUCTION

The isolated eigenvalues at the beginning of the spectrum of a self-adjoint operator can be characterized by the *maximum-minimum principle*, (1), which is based on the inequalities of Weyl [1], and by the *minimum-maximum principle*, (2), which is based on the inequalities of Poincaré [2]. The classical procedure to attain the corresponding maxima and minima in both cases consists of using subspaces spanned by eigenvectors. It is rarely mentioned that such "classical choices" are *only sufficient* and *not in general necessary*.

In the new maximum-minimum theory [3], [4] Weinstein characterized the class of subspaces for which the maximum of the minimum is attained, that is, for which equality holds in Weyl's inequality. Moreover, he showed that if a given eigenvalue of a compact self-adjoint operator is followed by a strictly greater eigenvalue, then there always exists a "nonclassical choice" and that for the largest eigenvalue of a symmetric matrix there is only the classical choice.

Later the present author [5] characterized the class of subspaces for which the minimum of the maximum is attained, that is, for which equality holds in Poincaré's inequality, and gave an example of a nonclassical choice.

The purpose of the present paper is to reformulate and extend to a class of unbounded operators Weinstein's results as a theorem which characterizes a class of operators for which there always exists a nonclassical choice in the maximum-minimum principle and to develop an analogous theorem for the existence of a nonclassical choice in the minimum-maximum principle. It turns out, somewhat unexpectedly, that there always exists a nonclassical choice, with some nearly trivial exceptions.

2. Definitions and Notations

Let $\mathscr{S}$ be the class of operators A such that A is a self-adjoint operator having a dense domain $\mathfrak{D}$ in a Hilbert space $\mathfrak{H}$ and the lower part of the spectrum of A consists of isolated eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \cdots$ each having finite multiplicity. We denote by $u_1, u_2, \ldots$, *any* orthonormal sequence of eigenvectors corresponding to $\lambda_1, \lambda_2, \ldots$. As usual let $R(u) = (Au, u)/(u, u)$, $u \neq 0$, $u \in \mathfrak{D}$ denote the Rayleigh quotient, let $R_\lambda = [A - \lambda I]^{-1}$ be the resolvent of A, and let E_λ be the resolution of the identity for A. Let $m(n) = \min\{j \mid \lambda_j = \lambda_n\}$ and $M(n) = \max\{j \mid \lambda_j = \lambda_n\}$, $(n = 1, 2, \ldots)$.

Let us now state the two variational principles which we recently extended to operators of class $\mathscr{S}$ [6].

The Maximum-Minimum Principle. *For a given index n $(n = 2, 3, \ldots)$, let $\mathfrak{P}_n$ denote any subspace of $\mathfrak{H}$ such that*

$$m(n) \leqslant \dim \mathfrak{P}_n + 1 \leqslant M(n).$$

Then λ_n is given by

$$\lambda_n = \max_{\mathfrak{P}_n} \min_{\substack{u \in \mathfrak{D} \\ u \perp \mathfrak{P}_n}} R(u). \tag{1}$$

The Minimum-Maximum Principle. *For a given index n $(n = 1, 2, \ldots)$, let $\mathfrak{B}_n$ denote any subspace of $\mathfrak{D}$ such that*

$$m(n) \leqslant \dim \mathfrak{B}_n \leqslant M(n).$$

Then λ_n is given by

$$\lambda_n = \min_{\mathfrak{B}_n} \max_{u \in \mathfrak{B}_n} R(u). \tag{2}$$

For noncompact operators, before our paper [6], principle (1) had always been formulated only as a sup-inf principle, see for instance [7, p. 127], [8, p. 1543], in which case the problem of nonclassical choices could not even be stated. Let us note in passing that an essential part of the proof of the existence of the minimum in (1) was our lemma[1] on self-adjoint operators [9], which has been generalized many times since then, see [10–14].

Moreover, let us emphasize as in [6] that if the eigenvalues are ordered in a descending sequence, the words maximum and minimum

[1] Lemma. *If A is self-adjoint and if Q is an orthogonal projection operator onto a subspace of finite codimension, then QAQ is self-adjoint.*

must be interchanged but (1) and (2) still define *two very different principles*, see also [15, p. 279]. According to our convention, we would treat the positive eigenvalues of a compact operator A by considering $-A$ instead of A.

In order to be precise in the following we define nonclassical choice for both (1) and (2).

DEFINITION 1. For a given index $n(n = 2, 3, \ldots)$ a subspace $\mathfrak{P}_n$ is said to be a *nonclassical choice for the maximum of the minimum in* (1) if

$$\mathfrak{P}_n \neq sp\{u_1, u_2, \ldots, u_j\}, \qquad (j = m(n) - 1, m(n), \ldots, M(n) - 1).$$

DEFINITION 2. For a given index $n(n = 1, 2, \ldots)$ a subspace $\mathfrak{V}_n$ is said to be a *nonclassical choice for the minimum of the maximum in* (2) if $\mathfrak{V}_n \not\subset sp\{u_1, u_2, \ldots, u_{M(n)}\}$.

3. THE EXISTENCE OF NONCLASSICAL CHOICES

We begin our discussion here by making an observation about two results of Weinstein. In particular, he showed in [3] that if A is a compact self-adjoint operator and if $\lambda_n < 0$ is such that there is an eigenvalue $\lambda_k > \lambda_n$, then there always exists a nonclassical choice for (1). Weinstein also showed in [16] that if λ_n is the greatest eigenvalue of a symmetric matrix then the maximum of the minimum in (1) is attained only for a classical choice, see also [17]. It is of importance to note here that we can reformulate these two results for compact self-adjoint operators in the following way.

THEOREM 1. *The only case in which there does not exist a nonclassical choice for the maximum of the minimum in* (1) *is when A is an operator on a finite-dimensional space and λ_n is its greatest eigenvalue.*

We shall now show that in view of the existence of the minimum [6] this Theorem is valid without change even for unbounded operators of class $\mathscr{S}$.

Proof. If for a fixed λ_n there is a point in the spectrum of A, not necessarily an eigenvalue, which is greater than λ_n, then our proof parallels that of Weinstein [3]. Since λ_n is isolated and the spectrum is closed, we choose the smallest such element, say ξ. Let $\mathfrak{U} = sp\{u_1, u_2, \ldots, u_{M(n)}\}$. In view of the existence of ξ, we know that

$\mathfrak{U} \neq \mathfrak{D}$. Since $\dim \mathfrak{U} < \infty$ and $\mathfrak{D}$ is dense, it follows from a lemma of Gokhberg–Krein [18] that there exists a vector $w \in \mathfrak{U}^{\perp} \cap \mathfrak{D}$, $(w, w) = 1$, see [19, p. 103]. If $\lambda_1 < \lambda_n$, we choose a real scalar α such that

$$0 < \alpha^2 \leqslant (\xi - \lambda_n)/(\lambda_n - \lambda_1)$$

and set

$$p_1 = (1 + \alpha^2)^{-1/2} (u_1 + \alpha w)$$

and

$$p_2 = u_2, p_3 = u_3, \ldots, p_{m(n)-1} = u_{m(n)-1}.$$

In order to apply Weinstein's criterion [4], extended to class $\mathscr{S}$ in [20], we need to compute $(R_\lambda p_i, p_k)$ $(i, k = 1, 2, \ldots, m(n) - 1)$ for all $\lambda = \lambda_n - \epsilon$, where $\epsilon > 0$ is sufficiently small. For all λ, $\lambda_1 < \lambda < \xi$, we have

$$(R_\lambda p_1, p_1) = (1 + \alpha^2)^{-1} \left[(\lambda_1 - \lambda)^{-1} + \alpha^2 \int_{\xi^-}^{\infty} (\mu - \lambda)^{-1} \, d(E_\mu w, w)\right]$$
$$\leqslant (1 + \alpha^2)^{-1} [(\lambda_1 - \lambda)^{-1} + (\xi - \lambda_n)/(\xi - \lambda)(\lambda_n - \lambda_1)].$$

Putting $\lambda = \lambda_n - \epsilon$ we obtain the inequalities,

$$(R_\lambda p_1, p_1) \leqslant (1 + \alpha^2)^{-1} [(\lambda_1 - \lambda_n + \epsilon)^{-1} + (\lambda_n - \lambda_1)^{-1}] < 0.$$

For $i = 2, 3, \ldots, m(n) - 1$ and $\lambda = \lambda_n - \epsilon$ we have $(R_\lambda p_i, p_i) = (\lambda_i - \lambda_n + \epsilon) < 0$. Therefore, the diagonal matrix $\{(R_\lambda p_i, p_k)\}$ is negative-definite. By the criterion of [4], the maximum in (1) is attained for the nonclassical choice $\mathfrak{P}_n = sp\{p_1, p_2, \ldots, p_{m(n)-1}\}$. If $\lambda_1 = \lambda_n$, it follows that any choice yields the maximum in (1), see Ref. [21]. The existence of ξ guarantees the existence of nonclassical choices, for instance $\mathfrak{P}_n = sp\{w\}$. We now consider the possibility that λ_n is the largest point in the spectrum of A. From now on our procedure is different from that in [16]. Since λ_n and all preceding eigenvalues (if any) are isolated and each has finite multiplicity, the operator A must be an operator on a finite-dimensional space, say $\mathfrak{H}_N$, where $N = \dim \mathfrak{H}_N$. Clearly we have

$$\lambda_n = \max_{u \in \mathfrak{H}_N} R(u).$$

Suppose that the subspace $\mathfrak{P}_n$ is a nonclassical choice for the maximum of the minimum in (1). Since λ_n may have multiplicity greater than one, we have $n \leqslant N$ and $\dim \mathfrak{P}_n \leqslant M(n) - 1 = N - 1$. Then for any $v \in \mathfrak{P}_n^{\perp}$, $v \neq 0$, we have

$$\lambda_n = \min_{u \in \mathfrak{P}_n^{\perp}} R(u) \leqslant R(v) \leqslant \max_{u \in \mathfrak{H}_N} R(u) = \lambda_n. \qquad (3)$$

Therefore, equality holds throughout (3). In particular, $R(v) = \max_{u \in \mathfrak{H}_N} R(u)$ so that v must satisfy the Euler equation

$$Av = \lambda_n v. \tag{4}$$

Letting $\mathfrak{E}_n$ denote the eigenspace of λ_n we see from (4) that $\mathfrak{P}_n{}^{\perp} \subset \mathfrak{E}_n$, which implies $\mathfrak{P}_n \supset \mathfrak{E}_n{}^{\perp}$. Therefore, $\mathfrak{P}_n = sp\{u_1, u_2, \ldots, u_j\}$ for some j, $m(n) - 1 \leqslant j \leqslant M(n) - 1$, which means that $\mathfrak{P}_n$ is *not* a nonclassical choice. This contradicts our hypothesis and completes the proof of the theorem.

We now turn our attention to the minimum-maximum principle (2) where the situation is substantially different.

THEOREM 2. *The only two cases in which there does not exist a non-classical choice for the minimum of the maximum in* (2) *are*

(i) *A is an operator on a finite-dimensional space and λ_n is its greatest eigenvalue*;

(ii) *$\lambda_n = \lambda_1$, even if $\mathfrak{H}$ is infinite-dimensional.*

Proof. As in the proof of Theorem 1 we begin by assuming the existence of the point ξ in the spectrum of A and we choose w as before. Letting $\eta = R(w)$, we note that $\lambda_n < \xi \leqslant \eta$. If $\lambda_1 < \lambda_n$ we choose β so that

$$0 < \beta^2 \leqslant (\lambda_n - \lambda_1)/(\eta - \lambda_n)$$

and set

$$v_1 = (1 + \beta^2)^{-1/2} (u_1 + \beta w)$$

and

$$v_2 = u_2, v_3 = u_3, \ldots, v_n = u_n.$$

Then we have

$$([A - \lambda_n I] v_1, v_1) = (1 + \beta^2)^{-1} [(\lambda_1 - \lambda_n) + \beta^2(\eta - \lambda_n)] \leqslant 0$$

and

$$([A - \lambda_n I] v_j, v_j) = \lambda_j - \lambda_n \leqslant 0 \qquad (j = 2, 3, \ldots, n).$$

The diagonal matrix $\{([A - \lambda_n I] v_i, v_j)\}$ is negative semidefinite and therefore, it follows from our criterion [6, p. 647] that the minimum in (2) is attained for the nonclassical choice $\mathfrak{V}_n = sp\{v_1, v_2, \ldots, v_n\}$. If λ_n is the greatest eigenvalue of an operator on a finite-dimensional space $\mathfrak{H}_N$, then $\mathfrak{H}_N = sp\{u_1, u_2, \ldots, u_{M(n)}\}$ so that nonclassical choices do not exist. On the other hand, if $A \in \mathscr{S}$ and $\lambda_1 = \lambda_n$, we assume that $\mathfrak{B}_n$ is a nonclassical choice for (2). Then we would have

$$\lambda_1 = \min_{u \in \mathfrak{D}} R(u) \leqslant \min_{u \in \mathfrak{B}_n} R(u) \leqslant \max_{u \in \mathfrak{B}_n} R(u) = \lambda_n ,$$

so that $\mathfrak{B}_n \subset \mathfrak{E}_n$, which contradicts our assumption that $\mathfrak{B}_n$ is nonclassical and completes the proof.

In view of Theorems 1 and 2 we see that nonclassical choices in both the maximum-minimum principle and the minimum-maximum principle are much more the rule than the exception.

4. Simultaneous Equalities

The conclusion in the case $\lambda_1 = \lambda_n$ in Theorem 2 is actually a special case of the following theorem which we announced in [5] and which we can apply here to an inequality of Fan [22].

Let us first briefly review Poincaré's inequalities which are the basis of (2) but do not appear there explicitly. We remark in passing that these inequalities for $r \geqslant 2$ are usually attributed without sufficient justification to Ritz, for more details see [23]. Let $\mathfrak{B}$ be an r-dimensional subspace of $\mathfrak{D}$, let V be the orthogonal projection operator onto $\mathfrak{B}$, and let $\Lambda_1 \leqslant \Lambda_2 \leqslant \cdots \leqslant \Lambda_r$ be those eigenvalues of VAV which correspond to eigenvectors $w_1, w_2, \ldots, w_r$ in $\mathfrak{B}$. Then we have Poincaré's inequalities

$$\lambda_1 \leqslant \Lambda_1 , \lambda_2 \leqslant \Lambda_2 , \ldots, \lambda_r \leqslant \Lambda_r . \tag{5}$$

Theorem 3. *The simultaneous equalities,*

$$\lambda_1 = \Lambda_1 , \lambda_2 = \Lambda_2 , \ldots, \lambda_r = \Lambda_r , \tag{6}$$

hold if, and only if,

$$\mathfrak{B} = sp\{u_1 , u_2 , \ldots, u_r\}. \tag{7}$$

Proof. The sufficiency of (7) is obvious, see [23], so let us suppose that (6) holds. We first note that

$$\lambda_1 = \min_{u \in \mathfrak{D}} R(u) \leqslant R(w_1) = \Lambda_1 = \lambda_1 .$$

Therefore, w_1 satisfies $Aw_1 = \lambda_1 w_1$. Using induction, we suppose that

$$Aw_k = \lambda_k w_k \qquad (k = 1, 2, \ldots, s < r).$$

Then we have

$$\lambda_{s+1} = \min_{\substack{u \in \mathfrak{D} \\ (u, w_i)=0 \\ (i=1,2,\ldots s)}} R(u) \leqslant R(w_{s+1}) = \Lambda_{s+1} = \lambda_{s+1},$$

which by the classical variational theory means that w_{s+1} satisfies $Aw_{s+1} = \lambda_{s+1} w_{s+1}$, and therefore, (7) holds.

In order to obtain a criterion for equality in an inequality of Fan [22], we give here a proof of the inequality which differs from the original proof of Fan and from subsequent proofs of Hersch [24] and Diaz–Metcalf [25]. Let $v_1, v_2, \ldots, v_r$ be any orthonormal set of vectors in $\mathfrak{D}$. Then we have Fan's inequality,

$$\lambda_1 + \lambda_2 + \cdots + \lambda_r \leqslant \sum_{i=1}^{r} R(v_i). \tag{8}$$

Proof. Since we have (5) and the invariance of the trace of an operator of finite rank we obtain the inequality,

$$\lambda_1 + \lambda_2 + \cdots + \lambda_r \leqslant \Lambda_1 + \Lambda_2 + \cdots + \Lambda_r = \sum_{i=1}^{r} R(w_i) = \sum_{i=1}^{r} R(v_i), \tag{9}$$

which yields (8).

Now the following criterion is an immediate consequence of Theorem 3 and (9).

THEOREM 4. *Equality holds in* (8) *if, and only if,*

$$sp\{v_1, v_2, \ldots, v_r\} = sp\{u_1, u_2, \ldots, u_r\}.$$

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